

Equilibrium thermodynamics in modified gravitational theories

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We show that it is possible to obtain a picture of equilibrium thermodynamics on the apparent horizon in the expanding cosmological background for a wide class of modified gravity theories with the Lagrangian density $f(R, \phi, X)$, where R is the Ricci scalar and X is the kinetic energy of a scalar field ϕ . This comes from a suitable definition of an energy momentum tensor of the “dark” component that respects to a local energy conservation in the Jordan frame. In this framework the horizon entropy S corresponding to equilibrium thermodynamics is equal to a quarter of the horizon area A in units of gravitational constant G , as in Einstein gravity. For a flat cosmological background with a decreasing Hubble parameter, S globally increases with time, as it happens for viable $f(R)$ inflation and dark energy models. We also show that the equilibrium description in terms of the horizon entropy S is convenient because it takes into account the contribution of both the horizon entropy \hat{S} in non-equilibrium thermodynamics and an entropy production term.

I. INTRODUCTION

The discovery of black hole entropy by Bekenstein opened up a window for a profound physical connection between gravity and thermodynamics [1]. The gravitational entropy S in Einstein gravity is proportional to the horizon area A of black holes, such that $S = A/(4G)$, where G is gravitational constant. A black hole with mass M obeys the first law of thermodynamics, $TdS = dM$ [2], where $T = |\kappa_s|/(2\pi)$ is a Hawking temperature determined by the surface gravity κ_s [3].

Since black hole solutions follow from Einstein field equations, the first law of black hole thermodynamics implies some connection between thermodynamics and Einstein equations. In fact Jacobson [4] showed that Einstein equations can be derived by using the Clausius relation $TdS = dQ$ on all local acceleration horizons in the Rindler space-time together with the relation $S \propto A$, where dQ and T are the energy flux across the horizon and the Unruh temperature seen by an accelerating observer just inside the horizon, respectively. This approach was applied to a number of cosmological settings such as quasi de Sitter inflationary universe [5, 6] and dark energy dominated universe [7].

Unlike stationary black holes the expanding universe has a dynamically changing apparent horizon. For dynamical black holes, Hayward [8] developed a framework to deal with the first law of thermodynamics on a trapping horizon in Einstein gravity (see Ref. [9] for related works). This was extended to the Friedmann-Lemaître-Robertson-Walker (FLRW) space-time [10], in which the Friedmann equation can be written in the form $TdS = -dE + WdV$, where E is the intrinsic energy and W is the work density present in the dynamical background. For matter contents of the universe with energy density ρ and pressure P , the work density is given by $W = (\rho - P)/2$. Note that the energy flux dQ in the Jacobson’s formalism is equivalent to $-dE + WdV$ in the FLRW background.

In the theories where the Lagrangian density f is a

non-linear function in terms of the Ricci scalar R (so called “ $f(R)$ gravity”), Eling *et al.* [11] pointed out that a non-equilibrium treatment is required such that the Clausius relation is modified to $d\hat{S} = dQ/T + d_i\hat{S}$. Here the horizon entropy is defined by $\hat{S} = F(R)A/(4G)$ with $F(R) = \partial f/\partial R$ and $d_i\hat{S}$ describes a bulk viscosity entropy production term. The variation of the quantity $F(R)$ gives rise to the non-equilibrium term $d_i\hat{S}$ that is absent in Einstein gravity, where a hat denotes the quantity in the non-equilibrium thermodynamics.

The connections between thermodynamics and modified gravity have been extensively discussed, including the theories such as $f(R)$ gravity [12–16], scalar-tensor theory [13–15, 17], Gauss-Bonnet and Lovelock gravity [17–19], and braneworld models [20] (see also Ref. [21]). In Gauss-Bonnet and Lovelock gravity, it is possible to rewrite Einstein equations in the form of equilibrium thermodynamics [17]. On the other hand, in $f(R)$ gravity and scalar-tensor theories with the Lagrangian density $F(\phi)R$, it was shown in Refs. [12, 14, 15, 17] that, by employing the Hayward’s dynamical framework, a non-equilibrium description of thermodynamics arises for the Wald’s horizon entropy defined by $\hat{S} = FA/(4G)$ [22, 23]. This is consistent with the results obtained by Eling *et al.* [11].

The appearance of a non-equilibrium entropy production term $d_i\hat{S}$ is intimately related to the theories in which the derivative of the Lagrangian density f with respect to R is not constant. It is of interest to see whether an equilibrium description of thermodynamics is possible in such modified gravity theories. In this paper we will show that equilibrium thermodynamics exists for the general Lagrangian density $f(R, \phi, X)$, where f is function of R , a scalar field ϕ , and a field kinetic energy X . This is possible by introducing the Bekenstein-Hawking entropy with a suitable redefinition of the “dark component” that respects a local energy conservation. This approach is convenient because the horizon entropy S defined in this framework involves the information of the

horizon entropy \hat{S} in non-equilibrium thermodynamics together with the entropy production term.

II. THERMODYNAMICS IN MODIFIED GRAVITY–NON-EQUILIBRIUM PICTURE

We start with the following action

$$I = \frac{1}{16\pi G} \int d^4x \sqrt{-g} f(R, \phi, X) - \int d^4x \mathcal{L}_M(g_{\mu\nu}, \Psi_M), \quad (1)$$

where g is the determinant of the metric tensor $g_{\mu\nu}$, \mathcal{L}_M is the matter Lagrangian that depends on $g_{\mu\nu}$ and matter fields Ψ_M , and $X = -(1/2)g^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi$ is the kinetic term of a scalar field ϕ (∇_μ is the covariant derivative operator associated with $g_{\mu\nu}$). The action (1) can describe a wide variety of modified gravity theories, e.g., $f(R)$ gravity, Brans-Dicke theories, scalar-tensor theories, and dilaton gravity, in addition to quintessence and k-essence.

From the action (1), the gravitational field equation and the equation of motion for ϕ are derived as [24]

$$FG_{\mu\nu} = 8\pi GT_{\mu\nu}^{(M)} + \frac{1}{2}g_{\mu\nu}(f - RF) + \nabla_\mu\nabla_\nu F - g_{\mu\nu}\square F + \frac{1}{2}f_{,X}\nabla_\mu\phi\nabla_\nu\phi, \quad (2)$$

$$\frac{1}{\sqrt{-g}}\partial_\mu(f_{,X}\sqrt{-g}g^{\mu\nu}\partial_\nu\phi) + f_{,\phi} = 0, \quad (3)$$

where $F \equiv \partial f/\partial R$, $f_{,X} \equiv \partial f/\partial X$, $f_{,\phi} \equiv \partial f/\partial\phi$, $T_{\mu\nu}^{(M)} = (2/\sqrt{-g})\delta\mathcal{L}_M/\delta g^{\mu\nu}$, and $G_{\mu\nu} = R_{\mu\nu} - (1/2)g_{\mu\nu}R$ is the Einstein tensor. For the matter energy momentum tensor $T_{\mu\nu}^{(M)}$ we consider perfect fluids of ordinary matter (radiation and non-relativistic matter) with total energy density ρ_f and pressure P_f .

We assume the 4-dimensional Friedmann-Lemaître-Robertson-Walker (FLRW) space-time with the metric,

$$ds^2 = h_{\alpha\beta}dx^\alpha dx^\beta + \bar{r}^2 d\Omega^2, \quad (4)$$

where $\bar{r} = a(t)r$ and $x^0 = t, x^1 = r$ with the 2-dimensional metric $h_{\alpha\beta} = \text{diag}(-1, a^2(t)/[1 - K\bar{r}^2])$. Here $a(t)$ is the scale factor, K is the cosmic curvature, and $d\Omega^2$ is the metric of 2-dimensional sphere with unit radius. In the background (4) we obtain the following field equations from Eqs. (2) and (3):

$$3F(H^2 + K/a^2) = f_{,X}X + \frac{1}{2}(FR - f) - 3H\dot{F} + 8\pi G\rho_f, \quad (5)$$

$$-2F(\dot{H} - K/a^2) = f_{,X}X + \ddot{F} - H\dot{F} + 8\pi G(\rho_f + P_f), \quad (6)$$

$$\frac{1}{a^3}(a^3\dot{\phi}f_{,X})' = f_{,\phi}, \quad (7)$$

where a dot represents a derivative with respect to cosmic time t and the scalar curvature is given by $R = 6(2H^2 +$

$\dot{H} + K/a^2)$. The perfect fluid satisfies the continuity equation

$$\dot{\rho}_f + 3H(\rho_f + P_f) = 0. \quad (8)$$

Equations (5) and (7) can be written as

$$H^2 + \frac{K}{a^2} = \frac{8\pi G}{3F}(\hat{\rho}_d + \rho_f), \quad (9)$$

$$\dot{H} - \frac{K}{a^2} = -\frac{4\pi G}{F}(\hat{\rho}_d + \hat{P}_d + \rho_f + P_f), \quad (10)$$

where

$$\hat{\rho}_d \equiv \frac{1}{8\pi G} \left[f_{,X}X + \frac{1}{2}(FR - f) - 3H\dot{F} \right], \quad (11)$$

$$\hat{P}_d \equiv \frac{1}{8\pi G} \left[\ddot{F} + 2H\dot{F} - \frac{1}{2}(FR - f) \right]. \quad (12)$$

We use a hat to represent quantities in the non-equilibrium description of thermodynamics. Note that $\hat{\rho}_d$ and \hat{P}_d originate from the energy-momentum tensor $\hat{T}_{\mu\nu}^{(d)}$ defined by

$$\hat{T}_{\mu\nu}^{(d)} \equiv \frac{1}{8\pi G} \left[\frac{1}{2}g_{\mu\nu}(f - RF) + \nabla_\mu\nabla_\nu F - g_{\mu\nu}\square F + \frac{1}{2}f_{,X}\nabla_\mu\phi\nabla_\nu\phi \right], \quad (13)$$

where the Einstein equation can be written as

$$G_{\mu\nu} = \frac{8\pi G}{F}(\hat{T}_{\mu\nu}^{(d)} + T_{\mu\nu}^{(M)}). \quad (14)$$

If we define the density $\hat{\rho}_d$ and the pressure \hat{P}_d of “dark” components in this way, we find that these obey the following equation

$$\dot{\hat{\rho}}_d + 3H(\hat{\rho}_d + \hat{P}_d) = \frac{3}{8\pi G}(H^2 + K/a^2)\dot{F}, \quad (15)$$

where we have used Eq. (7). For the theories with $\dot{F} \neq 0$ the r.h.s. of Eq. (15) does not vanish, so that the standard continuity equation does not hold. This happens for $f(R)$ gravity and scalar-tensor theory.

Let us proceed to the thermodynamical property of the theories given above. First of all the apparent horizon is determined by the condition $h^{\alpha\beta}\partial_\alpha\bar{r}\partial_\beta\bar{r} = 0$, which means that the vector $\nabla\bar{r}$ is null on the surface of the apparent horizon. For the FLRW space-time the radius of the apparent horizon is given by $\bar{r}_A = (H^2 + K/a^2)^{-1/2}$. Taking the time derivative of this relation and using Eq. (10) it follows that

$$\frac{F d\bar{r}_A}{4\pi G} = \bar{r}_A^3 H (\hat{\rho}_d + \hat{P}_d + \rho_f + P_f) dt. \quad (16)$$

The Bekenstein-Hawking horizon entropy in the Einstein gravity is given by $S = A/(4G)$, where $A = 4\pi\bar{r}_A^2$ is the area of the apparent horizon [1–3]. In the context of

modified gravity theories, Wald introduced a horizon entropy \hat{S} associated with a Noether charge [22, 23]. The Wald entropy \hat{S} is a local quantity defined in terms of quantities on the bifurcate Killing horizon. More specifically, it depends on the variation of the Lagrangian density of gravitational theories with respect to the Riemann tensor. This is equivalent to $\hat{S} = A/(4G_{\text{eff}})$, where $G_{\text{eff}} = G/F$ is the effective gravitational coupling [26]. Using the Wald entropy

$$\hat{S} = \frac{AF}{4G}, \quad (17)$$

together with Eq. (16), we obtain

$$\frac{1}{2\pi\bar{r}_A}d\hat{S} = 4\pi\bar{r}_A^3 H (\hat{\rho}_d + \hat{P}_d + \rho_f + P_f) dt + \frac{\bar{r}_A}{2G}dF. \quad (18)$$

The apparent horizon has the following Hawking temperature $T = |\kappa_s|/(2\pi)$, where κ_s is the surface gravity given by

$$\begin{aligned} \kappa_s &= -\frac{1}{\bar{r}_A} \left(1 - \frac{\dot{\bar{r}}_A}{2H\bar{r}_A} \right) = -\frac{\bar{r}_A}{2} \left(\dot{H} + 2H^2 + \frac{K}{a^2} \right) \\ &= -\frac{2\pi G}{3F} \bar{r}_A (\hat{\rho}_T - 3\hat{P}_T), \end{aligned} \quad (19)$$

with $\hat{\rho}_T = \hat{\rho}_d + \rho_f$ and $\hat{P}_T = \hat{P}_d + P_f$. As long as the total equation of state $w_T = \hat{P}_T/\hat{\rho}_T$ satisfies $w_T \leq 1/3$ it follows that $\kappa_s \leq 0$, which is the case for standard cosmology. Hence the horizon temperature is

$$T = \frac{1}{2\pi\bar{r}_A} \left(1 - \frac{\dot{\bar{r}}_A}{2H\bar{r}_A} \right). \quad (20)$$

Multiplying the term $1 - \dot{\bar{r}}_A/(2H\bar{r}_A)$ for Eq. (18), we obtain

$$\begin{aligned} Td\hat{S} &= 4\pi\bar{r}_A^3 H (\hat{\rho}_d + \hat{P}_d + \rho_f + P_f) dt \\ &\quad - 2\pi\bar{r}_A^2 (\hat{\rho}_d + \hat{P}_d + \rho_f + P_f) d\bar{r}_A + \frac{T}{G} \pi\bar{r}_A^2 dF. \end{aligned} \quad (21)$$

In Einstein gravity the Misner-Sharp energy [25] is defined to be $E = \bar{r}_A/(2G)$. In $f(R)$ gravity and scalar-tensor theory this was extended to the form $\hat{E} = \bar{r}_A F/(2G)$ [13]. Using this latter expression for $f(R, \phi, X)$ theories, it follows that

$$\hat{E} = \frac{\bar{r}_A F}{2G} = V \frac{3F(H^2 + K/a^2)}{8\pi G} = V(\hat{\rho}_d + \rho_f), \quad (22)$$

where $V = 4\pi\bar{r}_A^3/3$ is the volume inside the apparent horizon. Using Eqs. (8) and (15), we find the following relation

$$\begin{aligned} d\hat{E} &= -4\pi\bar{r}_A^3 H (\hat{\rho}_d + \hat{P}_d + \rho_f + P_f) dt \\ &\quad + 4\pi\bar{r}_A^2 (\hat{\rho}_d + \rho_f) d\bar{r}_A + \frac{\bar{r}_A}{2G} dF. \end{aligned} \quad (23)$$

From Eqs. (21) and (23) we obtain

$$\begin{aligned} Td\hat{S} &= -d\hat{E} + 2\pi\bar{r}_A^2 (\hat{\rho}_d + \rho_f - \hat{P}_d - P_f) d\bar{r}_A \\ &\quad + \frac{\bar{r}_A}{2G} (1 + 2\pi\bar{r}_A T) dF. \end{aligned} \quad (24)$$

As in Refs. [8, 17] we introduce the work density

$$\hat{W} = (\hat{\rho}_d + \rho_f - \hat{P}_d - P_f)/2. \quad (25)$$

Then Eq. (24) reduces to

$$Td\hat{S} = -d\hat{E} + \hat{W}dV + \frac{\bar{r}_A}{2G} (1 + 2\pi\bar{r}_A T) dF. \quad (26)$$

This equation can be written in the form

$$Td\hat{S} + Td_i\hat{S} = -d\hat{E} + \hat{W}dV, \quad (27)$$

where

$$d_i\hat{S} = -\frac{1}{T} \frac{\bar{r}_A}{2G} (1 + 2\pi\bar{r}_A T) dF = -\left(\frac{\hat{E}}{T} + \hat{S} \right) \frac{dF}{F}. \quad (28)$$

The new term $d_i\hat{S}$ may be interpreted as a term of entropy production in the non-equilibrium thermodynamics. The theories with $F = \text{constant}$ lead to $d_i\hat{S} = 0$, which means that the first-law of the equilibrium thermodynamics holds. Meanwhile the theories with $dF \neq 0$, including $f(R)$ gravity and scalar-tensor theory, give rise to the additional term (28).

It is clear from Eq. (15) that the density $\hat{\rho}_d$ and the pressure \hat{P}_d defined in Eqs. (11) and (12) do not satisfy the standard continuity equation for $\dot{F} \neq 0$. If it is possible to define $\hat{\rho}_d$ and \hat{P}_d obeying the conserved equation, then we anticipate that the non-equilibrium description of thermodynamics may not be necessary. In the next section we shall show that such a treatment is indeed possible.

III. EQUILIBRIUM INTERPRETATION OF THERMODYNAMICS IN MODIFIED GRAVITY

Let us consider equilibrium description of thermodynamics for theories with the action (1). Equations (5) and (6) can be written as

$$3 \left(H^2 + \frac{K}{a^2} \right) = 8\pi G (\rho_d + \rho_f), \quad (29)$$

$$-2 \left(\dot{H} - \frac{K}{a^2} \right) = 8\pi G (\rho_d + P_d + \rho_f + P_f), \quad (30)$$

where

$$\rho_d \equiv \frac{1}{8\pi G} \left[f_{,X} X + \frac{1}{2}(FR - f) - 3H\dot{F} + 3(1-F)(H^2 + K/a^2) \right], \quad (31)$$

$$P_d \equiv \frac{1}{8\pi G} \left[\ddot{F} + 2H\dot{F} - \frac{1}{2}(FR - f) - (1-F)(2\dot{H} + 3H^2 + K/a^2) \right]. \quad (32)$$

If we define ρ_d and P_d in this way, they obey the following continuity equation

$$\dot{\rho}_d + 3H(\rho_d + P_d) = 0, \quad (33)$$

where we have used Eq. (7). We then find that Eq. (16) is replaced by

$$\frac{d\bar{r}_A}{4\pi G} = \bar{r}_A^3 H (\rho_d + P_d + \rho_f + P_f) dt. \quad (34)$$

In the equilibrium description there is no need to define G_{eff} in the field equations in (29) and (30) in contrast to Eqs. (9) and (10) in the non-equilibrium description. This originates from the redefinition of energy density ρ_d and pressure P_d of “dark” components defined in Eqs. (31) and (32), respectively. This redefinition leads to the continuity equation (33). In other words, the energy-momentum conservation in terms of “dark” components is met. Since the perfect fluid of ordinary matter also satisfies the continuity equation (8), the total energy density $\rho_T \equiv \rho_d + \rho_f$ and the total pressure $P_T \equiv P_d + P_f$ of the universe obey the continuity equation

$$\dot{\rho}_T + 3H(\rho_T + P_T) = 0. \quad (35)$$

Hence the equilibrium treatment of thermodynamics can be executed similarly to that in Einstein gravity. As a consequence, we introduce the Bekenstein-Hawking entropy [1–3]

$$S = \frac{A}{4G} = \frac{\pi}{G} \frac{1}{H^2 + K/a^2}, \quad (36)$$

unlike the Wald entropy associated with G_{eff} in Eq. (17) in the non-equilibrium thermodynamics. This allows us to obtain the equilibrium description of thermodynamics as that in Einstein gravity. Note that the Bekenstein-Hawking entropy (36) is a global geometric quantity proportional to A , which is not directly affected by the difference of gravitational theories (i.e. by the difference of the quantity $F = \partial f / \partial R$).

From the definition (36) it follows that

$$\frac{1}{2\pi\bar{r}_A} dS = 4\pi\bar{r}_A^3 H (\rho_d + P_d + \rho_f + P_f) dt. \quad (37)$$

Using the horizon temperature given in Eq. (20), we obtain

$$TdS = 4\pi\bar{r}_A^3 H (\rho_d + P_d + \rho_f + P_f) dt - 2\pi\bar{r}_A^2 (\rho_d + P_d + \rho_f + P_f) d\bar{r}_A. \quad (38)$$

Defining the Misner-Sharp energy to be

$$E = \frac{\bar{r}_A}{2G} = V(\rho_d + \rho_f), \quad (39)$$

we find

$$dE = -4\pi\bar{r}_A^3 H (\rho_d + P_d + \rho_f + P_f) dt + 4\pi\bar{r}_A^2 (\rho_d + \rho_f) d\bar{r}_A. \quad (40)$$

Due to the conservation equation (33), the r.h.s. of Eq. (40) does not include an additional term proportional to dF . Combining Eqs. (38) and (40) gives

$$TdS = -dE + WdV, \quad (41)$$

where the work density W is defined by

$$W = (\rho_d + \rho_f - P_d - P_f) / 2. \quad (42)$$

Equation (41) corresponds to the first law of equilibrium thermodynamics. This shows that the equilibrium form of thermodynamics can be derived by introducing the density ρ_d and the pressure P_d in a suitable way.

Plugging Eqs. (39) and (42) into Eq. (41), we find

$$T\dot{S} = V \left(3H - \frac{\dot{V}}{2V} \right) (\rho_d + \rho_f + P_d + P_f). \quad (43)$$

Using $V = 4\pi\bar{r}_A^3/3$ and Eq. (20), it follows that

$$\begin{aligned} \dot{S} &= 6\pi H V \bar{r}_A (\rho_d + \rho_f + P_d + P_f) \\ &= -\frac{2\pi}{G} \frac{H(\dot{H} - K/a^2)}{(H^2 + K/a^2)^2}. \end{aligned} \quad (44)$$

The horizon entropy increases as long as the null energy condition $\rho_T + P_T = \rho_d + \rho_f + P_d + P_f \geq 0$ is satisfied. S decreases for the total equation of state $w_T \equiv P_T/\rho_T < -1$, as it happens in General Relativity. Meanwhile the Wald entropy (17) does not in general possess this property, as we will see for $f(R)$ inflation models in Sec. V A.

The above equilibrium picture of thermodynamics is intimately related with the fact that there is an energy momentum tensor $T_{\mu\nu}^{(d)}$ satisfying the local conservation law $\nabla^\mu T_{\mu\nu}^{(d)} = 0$. This corresponds to writing the Einstein equation in the form

$$G_{\mu\nu} = 8\pi G \left(T_{\mu\nu}^{(d)} + T_{\mu\nu}^{(M)} \right), \quad (45)$$

where

$$\begin{aligned} T_{\mu\nu}^{(d)} &\equiv \frac{1}{8\pi G} \left[\frac{1}{2} g_{\mu\nu} (f - R) + \nabla_\mu \nabla_\nu F - g_{\mu\nu} \square F \right. \\ &\quad \left. + \frac{1}{2} f_{,X} \nabla_\mu \phi \nabla_\nu \phi + (1-F) R_{\mu\nu} \right]. \end{aligned} \quad (46)$$

Defining $T_{\mu\nu}^{(d)}$ in this way, the local conservation of $T_{\mu\nu}^{(d)}$ follows from Eq. (45) because of the relations $\nabla^\mu G_{\mu\nu} = 0$ and $\nabla^\mu T_{\mu\nu}^{(M)} = 0$.

One can show that the horizon entropy S in the equilibrium picture has the following relation with \hat{S} in the non-equilibrium picture:

$$dS = d\hat{S} + d_i\hat{S} + \frac{\bar{r}_A}{2GT}dF - \frac{2\pi(1-F)}{G} \frac{H(\dot{H} - K/a^2)}{(H^2 + K/a^2)^2} dt. \quad (47)$$

Using the relations (28) and (37), Eq. (47) reduces to the following form

$$dS = \frac{1}{F}d\hat{S} + \frac{1}{F} \frac{2H^2 + \dot{H} + K/a^2}{4H^2 + \dot{H} + 3K/a^2} d_i\hat{S}, \quad (48)$$

where

$$d_i\hat{S} = -\frac{6\pi}{G} \frac{4H^2 + \dot{H} + 3K/a^2}{H^2 + K/a^2} \frac{dF}{R}. \quad (49)$$

While S is identical to \hat{S} in Einstein gravity ($F = 1$), the difference appears in modified gravity theories with $dF \neq 0$. Equation (48) shows that the change of the horizon entropy S in the equilibrium framework involves the information of both $d\hat{S}$ and $d_i\hat{S}$ in the non-equilibrium framework.

IV. EINSTEIN FRAME IN SCALAR-TENSOR THEORIES

For some specific theories, the action (1) can be transformed to the so-called Einstein frame via the conformal transformation. For example let us consider the following scalar-tensor theories with the action

$$I = \int d^4x \sqrt{-g} \left[\frac{F(\phi)}{2\kappa^2} R + \omega(\phi) X - V(\phi) \right] - \int d^4x \mathcal{L}_M(g_{\mu\nu}, \Psi_M), \quad (50)$$

where $\kappa^2 \equiv 8\pi G$, and $F(\phi)$, $\omega(\phi)$, $V(\phi)$ are functions of ϕ . Under the conformal transformation, $\tilde{g}_{\mu\nu} = Fg_{\mu\nu}$, we obtain the following action in the Einstein frame [27, 28]:

$$I_E = \int d^4x \sqrt{-\tilde{g}} \left[\frac{1}{2\kappa^2} \tilde{R} - \frac{1}{2} (\tilde{\nabla}\varphi)^2 - U(\varphi) \right] - \int d^4x \mathcal{L}_M(F(\phi)^{-1} \tilde{g}_{\mu\nu}, \Psi_M), \quad (51)$$

where a tilde represents quantities in the Einstein frame, and

$$\varphi \equiv \int d\phi \sqrt{\frac{3}{2} \left(\frac{F_{,\phi}}{\kappa F} \right)^2 + \frac{\omega}{F}}, \quad U = \frac{V}{F^2}. \quad (52)$$

Varying the action (51) with respect to φ , we obtain the field equation

$$\square\varphi - U_{,\varphi} - \frac{1}{\sqrt{-\tilde{g}}} \frac{\partial \mathcal{L}_M}{\partial \varphi} = 0. \quad (53)$$

In the FLRW background the following relations hold

$$d\tilde{t} = \sqrt{F} dt, \quad \tilde{a} = \sqrt{F} a. \quad (54)$$

Using the relation $-\partial \mathcal{L}_M / \partial \varphi = \sqrt{-\tilde{g}} \kappa Q(\varphi) \tilde{T}_M$, where $Q(\varphi) \equiv -F_{,\varphi} / (2\kappa F)$ and $\tilde{T}_M \equiv -\tilde{\rho}_f + 3\tilde{P}_f$, the field equation (53) reduces to

$$\ddot{\varphi} + 3\tilde{H}\dot{\varphi} + U_{,\varphi} = -Q(\varphi)(\tilde{\rho}_f - 3\tilde{P}_f). \quad (55)$$

In this section a dot represents a derivative with respect to \tilde{t} . Introducing $\tilde{\rho}_\varphi \equiv \dot{\varphi}^2/2 + U(\varphi)$ and $\tilde{P}_\varphi \equiv \dot{\varphi}^2/2 - U(\varphi)$, Eq. (55) can be written as

$$\dot{\tilde{\rho}}_\varphi + 3\tilde{H}(\tilde{\rho}_\varphi + \tilde{P}_\varphi) = -Q(\varphi)(\tilde{\rho}_f - 3\tilde{P}_f)\dot{\varphi}. \quad (56)$$

Note that $\tilde{\rho}_f$ and \tilde{P}_f in the Einstein frame are related with ρ_f and P_f in the Jordan frame via $\rho_f = F^2\tilde{\rho}_f$ and $P_f = F^2\tilde{P}_f$. Using Eqs. (8) and (54), it then follows that

$$\dot{\tilde{\rho}}_f + 3\tilde{H}(\tilde{\rho}_f + \tilde{P}_f) = +Q(\varphi)(\tilde{\rho}_f - 3\tilde{P}_f)\dot{\varphi}. \quad (57)$$

Equations (56) and (57) show that the field φ is coupled to matter with the coupling $Q(\varphi)$ except for radiation [29]. In Brans-Dicke theory with $F(\phi) = \kappa\phi$ and $\omega(\phi) = \omega_{BD}/(\kappa\phi)$ (ω_{BD} is a constant parameter), the coupling Q is a constant [28]. The $f(R)$ gravity in the metric formalism corresponds to $\omega_{BD} = 0$ and $V = (FR - f)/(2\kappa^2)$, in which case $\kappa\varphi = \sqrt{3/2} \ln F$ from Eq. (52). Hence the coupling $Q(\varphi) = -F_{,\varphi}/(2\kappa F)$ is also a constant ($Q = -1/\sqrt{6}$) in metric $f(R)$ gravity [36].

From Eqs. (56) and (57) the total energy density $\tilde{\rho}_T = \tilde{\rho}_\varphi + \tilde{\rho}_f$ and the total pressure $\tilde{P}_T = \tilde{P}_\varphi + \tilde{P}_f$ satisfy the continuity equation, $d\tilde{\rho}_T/d\tilde{t} + 3\tilde{H}(\tilde{\rho}_T + \tilde{P}_T) = 0$. The following equations also hold in the Einstein frame

$$3(\tilde{H}^2 + K/\tilde{a}^2) = \kappa^2(\tilde{\rho}_\varphi + \tilde{\rho}_f), \quad (58)$$

$$2(\tilde{H} - K/\tilde{a}^2) = -\kappa^2(\tilde{\rho}_\varphi + \tilde{P}_\varphi + \tilde{\rho}_f + \tilde{P}_f). \quad (59)$$

We define several thermodynamical quantities

$$\tilde{T} = \frac{|\tilde{\kappa}_s|}{2\pi}, \quad \tilde{S} = \frac{\tilde{A}}{4G}, \quad \tilde{E} = \frac{\tilde{r}_A}{2G}, \quad \tilde{V} = 4\pi\tilde{r}_A^3/3, \quad \tilde{W} = (\tilde{\rho}_\varphi + \tilde{\rho}_f - \tilde{P}_\varphi - \tilde{P}_f)/2, \quad (60)$$

where $\tilde{r}_A = (\tilde{H}^2 + K/\tilde{a}^2)^{-1/2}$, $\tilde{A} = 4\pi\tilde{r}_A^2$, and $\tilde{\kappa}_s = -[1 - (d\tilde{r}_A/d\tilde{t})/(2\tilde{H}\tilde{r}_A)]/\tilde{r}_A$. Following the similar procedure as in Sec. III, we arrive at the first law of thermodynamics,

$$\tilde{T}d\tilde{S} = -d\tilde{E} + \tilde{W}d\tilde{V}. \quad (61)$$

Hence the equilibrium description of thermodynamics holds in the Einstein frame.

In the following we shall find the relation of the equilibrium description in the Jordan frame for the flat universe

($K = 0$). The horizon entropy and the Misner-Sharp energy in the Einstein frame are given, respectively, by

$$\tilde{S} = \frac{\pi}{G} \frac{1}{\tilde{H}^2}, \quad \tilde{E} = \frac{1}{2G} \frac{1}{\tilde{H}}. \quad (62)$$

These are different from those in the equilibrium picture in the Jordan frame ($S = \pi/(GH^2)$ and $E = 1/(2GH)$) because of the difference of the Hubble parameter:

$$H = \sqrt{F} \left(\tilde{H} - \frac{1}{2F} \frac{dF}{dt} \right) = \sqrt{F} \left(\tilde{H} + \kappa Q \frac{d\varphi}{dt} \right). \quad (63)$$

In the Jordan frame the following relations hold

$$\begin{aligned} dE &= -\frac{dH}{2GH^2}, \quad WdV = \left(3 + \frac{1}{H^2} \frac{dH}{dt} \right) dE, \\ TdS &= \left(2 + \frac{1}{H^2} \frac{dH}{dt} \right) dE. \end{aligned} \quad (64)$$

Note that similar relations hold in the Einstein frame by adding the tilde for corresponding quantities. Using Eq. (63) we obtain

$$dE = (\mu/\sqrt{F})d\tilde{E}, \quad (65)$$

where

$$\mu \equiv \frac{1 + \kappa Q[\ddot{\varphi} + (Q_{,\varphi}/Q - \kappa Q)\dot{\varphi}^2 - \tilde{H}\dot{\varphi}]/\dot{\tilde{H}}}{(1 + \kappa Q\dot{\varphi}/\tilde{H})^2}, \quad (66)$$

and

$$WdV = \frac{\mu(3 + \mu\dot{\tilde{H}}/\tilde{H}^2)}{\sqrt{F}(3 + \dot{\tilde{H}}/\tilde{H}^2)} \tilde{W}d\tilde{V}, \quad (67)$$

$$TdS = \frac{\mu(2 + \mu\dot{\tilde{H}}/\tilde{H}^2)}{\sqrt{F}(2 + \dot{\tilde{H}}/\tilde{H}^2)} \tilde{T}d\tilde{S}. \quad (68)$$

Recall that the field φ in Eq. (66) satisfies Eq. (55).

Since $F = 1$ and $\mu = 1$ in Einstein gravity, one has $dE = d\tilde{E}$, $WdV = \tilde{W}d\tilde{V}$, and $TdS = \tilde{T}d\tilde{S}$. In scalar-tensor theories in which F and μ dynamically change in time, the equilibrium description of thermodynamics in the Jordan frame is not identical to that in the Einstein frame. We note that general modified gravity theories with the action (1) do not necessarily have the action in the Einstein frame. Even for such general theories we have shown in Sec. III that the equilibrium picture of thermodynamics is present without any reference to the Einstein frame.

Moreover, we regard that the frame in which the baryons obey the standard continuity equation $\rho_m \propto a^{-3}$, i.e. the Jordan frame, is the “physical” frame where physical quantities are compared with observations and experiments. The direct construction of the equilibrium thermodynamics in the Jordan frame is not only versatile but is physically well motivated. Only for the theories in which the action in the Einstein frame exists we can find the relation between the thermodynamical quantities in the two frames, as we have done above.

V. APPLICATION TO $f(R)$ THEORIES

In this section we apply the formulas of the horizon entropies in the Jordan frame to inflation and dark energy in $f(R)$ theories. In particular the evolution of S and \hat{S} will be discussed during inflation (and reheating) in $f(R)$ theories. We also study how the horizon entropies evolve during an epoch of the late-time cosmic acceleration in $f(R)$ dark energy models. In the following we assume the flat FLRW space-time ($K = 0$).

A. Inflation

It is known that cosmological inflation can be realized by the Lagrangian density of the form $f(R) = R + \alpha R^n$ ($\alpha, n > 0$). The first inflation model proposed by Starobinsky corresponds to $n = 2$ [30]. Let us consider the model $f(R) = R + \alpha R^n$ in the region $F = df/dR = 1 + n\alpha R^{n-1} \gg 1$.

During inflation one can use the approximations $|\dot{H}/H^2| \ll 1$ and $|\ddot{H}/(H\dot{H})| \ll 1$. With these approximations, in the absence of matter fluids Eq. (29) reduces to

$$\frac{\dot{H}}{H^2} = -\beta, \quad \beta \equiv \frac{2-n}{(n-1)(2n-1)}. \quad (69)$$

This gives the power-law evolution of the scale factor ($a \propto t^{1/\beta}$), which means that inflation occurs for $\beta < 1$, i.e. $n > (1 + \sqrt{3})/2$. When $n = 2$ one has $\beta = 0$, so that H is constant in the regime $F \gg 1$. The models with $n > 2$ lead to the super-inflation characterized by $\dot{H} > 0$ and $a \propto |t_0 - t|^{-1/|\beta|}$ (t_0 is a constant).

The standard inflation with decreasing H occurs for $0 < \beta < 1$, i.e. $(1 + \sqrt{3})/2 < n < 2$. In this case the horizon entropy (36) in the equilibrium framework grows as $S \propto H^{-2} \propto t^2$ during inflation. Meanwhile the horizon entropy $\hat{S} = F(R)A/(4G)$ in the non-equilibrium framework has a dependence $\hat{S} \propto R^{n-1}/H^2 \propto H^{2(n-2)} \propto t^{2(2-n)}$ in the regime $F \gg 1$. Hence \hat{S} grows more slowly relative to S . This property can be understood from Eq. (48), i.e.

$$\frac{dS}{dt} = \frac{1}{F} \frac{d\hat{S}}{dt} + \frac{1}{F} \frac{2-\beta}{4-\beta} \frac{d_i \hat{S}}{dt}, \quad (70)$$

where

$$\frac{d_i \hat{S}}{dt} = \frac{12\pi\beta(4-\beta)}{G} H F_{,R}. \quad (71)$$

Here $F_{,R} \equiv dF/dR$. For the above model the term $F = n\alpha R^{n-1}$ evolves as $F \propto t^{2(1-n)}$. This means that $(1/F)d\hat{S}/dt \propto t$ in Eq. (70), which has the same dependence as the time-derivative of S , i.e. $dS/dt \propto t$. The r.h.s. of Eq. (71) is positive because $F_{,R} > 0$ and $\beta > 0$, so that $d_i \hat{S}/dt > 0$. We have $d_i \hat{S}/dt \propto t^{3-2n}$ and hence

the last term on the r.h.s. of Eq. (70) also grows in proportion to t . Thus S evolves differently from \hat{S} because of the presence of the term $1/F$.

More precisely each term in Eq. (70) is given by

$$\frac{dS}{dt} = \frac{2\pi\beta}{G} \frac{1}{H}, \quad (72)$$

$$\frac{1}{F} \frac{d\hat{S}}{dt} = \frac{2\pi\beta(2-n)}{G} \frac{1}{H}, \quad (73)$$

$$\frac{1}{F} \frac{2-\beta}{4-\beta} \frac{d_i\hat{S}}{dt} = \frac{2\pi\beta(n-1)}{G} \frac{1}{H}, \quad (74)$$

where we have used $F \gg 1$. When $(1 + \sqrt{3})/2 < n < 2$, i.e. $0 < \beta < 1$, it follows that $dS > 0, d\hat{S} > 0, d_i\hat{S} > 0$ for $dt > 0$. In the limit that $n \rightarrow 2$ the ratio $r = (2-n)/(n-1)$ of the r.h.s. of Eq. (73) to the r.h.s. of Eq. (74) approaches 0, so that the entropy production term $d_i\hat{S}$ gives a dominant contribution to $d\hat{S}$. When $n > 2$, i.e. $\beta < 0$, we have $dS < 0, d\hat{S} > 0, d_i\hat{S} < 0$ for $dt > 0$. Hence the decrease of S comes from the negative entropy production term $d_i\hat{S}$. The entropy S in the equilibrium framework decreases for the theories with $\dot{H} > 0$, whereas \hat{S} in the non-equilibrium framework can grow even in such cases unless the entropy production term is taken into account. The equilibrium description of thermodynamics allows us to introduce the Bekenstein-Hawking entropy that mimics the property in General Relativity.

In the Starobinsky's model $f(R) = R + R^2/(6M^2)$, the presence of the linear term in R eventually causes inflation to end. Without neglecting this linear term, we obtain the following equations:

$$\ddot{H} - \frac{\dot{H}^2}{2H} + 3H\dot{H} + \frac{1}{2}M^2H = 0, \quad (75)$$

$$\ddot{R} + 3H\dot{R} + M^2R = 0. \quad (76)$$

During inflation the first two terms in Eq. (75) can be neglected relative to others, which gives $\dot{H} \simeq -M^2/6$. We then obtain the solution $H \simeq H_i - (M^2/6)(t - t_i)$ with the Ricci scalar $R \simeq 12H^2 - M^2$, where H_i is the Hubble parameter at the onset of inflation (at $t = t_i$). The accelerated expansion ends when the slow-roll parameter $\epsilon \equiv -\dot{H}/H^2 \simeq M^2/(6H^2)$ grows to the order of unity, i.e. $H \simeq M/\sqrt{6}$. The horizon entropy S grows as $S \propto [H_i - (M^2/6)(t - t_i)]^{-2}$, whereas $\hat{S} \simeq (\pi/G)[2/(3H^2) + 4/M^2] \approx 4\pi/(GM^2)$ during inflation ($H^2 \gg M^2$). Hence the horizon entropy S in the equilibrium framework increases faster than \hat{S} in the non-equilibrium one, as in the models with $(1 + \sqrt{3})/2 < n < 2$. This property is clearly seen in the numerical simulations of Fig. 1 ($0 < Mt < 30$).

As long as $H^2 \gg M^2$ both dS and the last term on the r.h.s. of Eq. (48) are approximately given by $\pi M^2/(3GH^3)dt$. Meanwhile we have $(1/F)d\hat{S} = \pi M^4/(18GH^5)dt$, which is suppressed by a factor of $M^2/(6H^2)$ relative to dS . Hence the variation of the

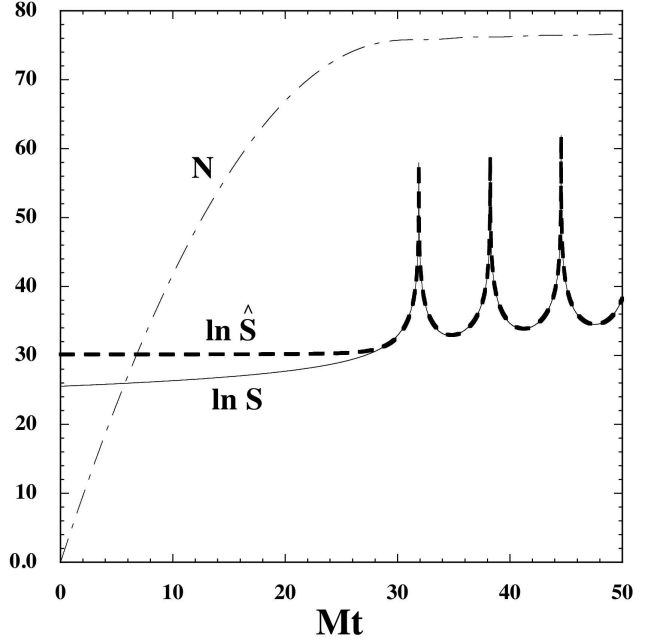


FIG. 1: The variation of the horizon entropy $S = A/(4G)$ in the equilibrium picture as well as $\hat{S} = FA/(4G)$ in the non-equilibrium picture for the inflation model $f(R) = R + R^2/(6M^2)$. We also show the evolution of the number of e-foldings $N = \ln(a/a_i)$ from the beginning of inflation. Initial conditions are chosen to be $R/M^2 = 305$ and $\dot{R} = 0$. The entropy S evolves faster than \hat{S} during inflation ($0 < Mt < 30$), but their evolution is similar after inflation in which F is close to 1.

horizon entropy S is mainly sourced by the entropy production term $d_i\hat{S}$ during inflation.

The inflationary period is followed by a reheating phase in which the Ricci scalar R exhibits a damped oscillation with a frequency M [see Eq. (76)]. The evolution of the Hubble parameter during the reheating period can be estimated as [31]

$$H \simeq \left[\frac{3}{M} + \frac{3}{4}(t - t_{\text{os}}) + \frac{3}{4M} \sin M(t - t_{\text{os}}) \right]^{-1} \times \cos^2 \left[\frac{M}{2}(t - t_{\text{os}}) \right], \quad (77)$$

where t_{os} is the time at which H starts to oscillate. Taking the time average of the oscillations in the region $M(t - t_{\text{os}}) \gg 1$ it follows that $\langle H \rangle \simeq (2/3)(t - t_{\text{os}})^{-1}$ and hence the Universe evolves as a matter-dominated one ($a \propto (t - t_{\text{os}})^{2/3}$).

The quantity $F = 1 + R/(3M^2)$ approaches 1 after the end of inflation. Equations (29) and (30) correspond to the standard Friedmann equations during the radiation and matter eras. S approaches \hat{S} after the end of inflation ($F \simeq 1$), which can be confirmed in the numerical simulation of Fig. 1. In this regime the entropy production term $(1/F)[(2-\beta)/(4-\beta)]d_i\hat{S}$ can be negligible relative to dS , so that $dS \simeq d\hat{S}$. There are intervals in

which the horizon entropies decrease because of the oscillation of H , but the important point is that both S and \hat{S} globally increase in proportion to $\langle H \rangle^{-2} \propto (t - t_{\text{os}})^2$.

The entropy production term $d_i \hat{S}$ is a dominant contribution to dS during inflation, but after inflation it begins to be suppressed relative to dS by a factor of the order $\langle H \rangle^2 / M^2 (\ll 1)$. This shows that it is more convenient to take the equilibrium framework in terms of the single horizon entropy S rather than the non-equilibrium framework that separates the horizon entropy into two contributions.

B. Dark energy

Let us next proceed to $f(R)$ dark energy models consistent with both cosmological and local gravity constraints. We focus on models in which cosmological solutions have a late-time de Sitter attractor at $R = R_1 (> 0)$ satisfying the condition $Rf_{,R} = 2f$. For the stability of the de Sitter point we require that [32–34]

$$0 < m(R_1) < 1, \quad m(R) \equiv \frac{Rf_{,RR}}{f_{,R}}, \quad (78)$$

where $f_{,RR} \equiv d^2 f / dR^2$. Here the quantity m characterizes the deviation from the Λ CDM model ($f(R) = R - 2\Lambda$).

There are several other conditions that viable $f(R)$ dark energy models need to satisfy: (i) $f_{,R} > 0$ for $R \geq R_1$ to avoid ghosts, (ii) $f_{,RR} > 0$ for $R \geq R_1$ to ensure the stability of cosmological perturbations [35] and to realize a matter-dominated epoch followed by the late-time cosmic acceleration [36], (iii) m rapidly approaches $+0$ for $R \gg R_0$ (R_0 is the cosmological Ricci scalar today) to satisfy local gravity constraints [37]. In other words the models need to be close to the Λ CDM model in the region $R \gg R_0$. More precisely, we require that $m(R) \lesssim 10^{-15}$ for $R \approx 10^5 R_0$ [28, 38].

A number of authors proposed viable models consistent with the above requirements [34, 39–46]. One representative model is [42]

$$f(R) = R - \lambda R_c \left[1 - \left(1 + \frac{R^2}{R_c^2} \right)^{-n} \right], \quad (79)$$

where λ , R_c , and n are positive constants. For $\lambda = \mathcal{O}(1)$, R_c is of the order of R_0 . Another similar model is $f(R) = R - \lambda R_c (R/R_c)^{2n} / [(R/R_c)^{2n} + 1]$ [41], which has the same asymptotic form $f(R) \simeq R - \lambda R_c [1 - (R/R_c)^{-2n}]$ as that in the model (79). When $n > 0.9$ these models are consistent with local gravity constraints due to the rapid decrease of m for increasing R [47].

A simpler $f(R)$ model that has only two free parameters λ and R_c is [44]

$$f(R) = R - \lambda R_c \tanh \left(\frac{R}{R_c} \right), \quad (80)$$

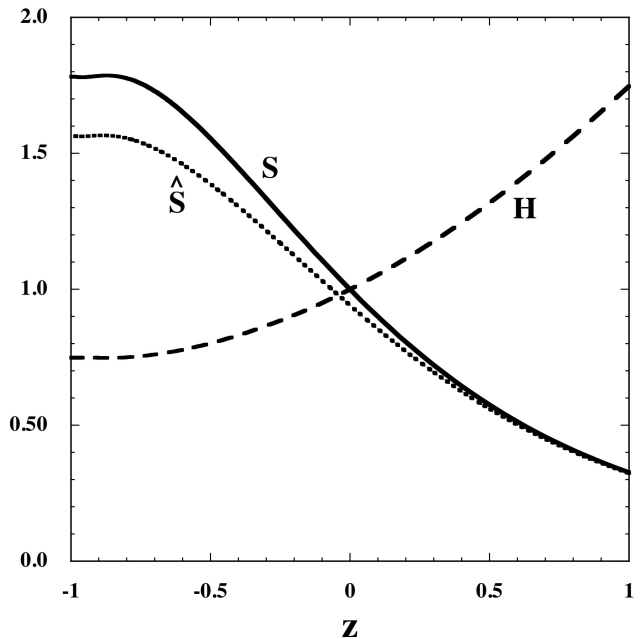


FIG. 2: The evolution of the horizon entropies $S = A/(4G)$ and $\hat{S} = FA/(4G)$ versus the redshift z for the dark energy model (79) with model parameters $n = 1$ and $\lambda = 1.8$. The entropy S is normalized to be 1 today ($z = 0$). The initial evolution of S is similar to that of \hat{S} , but the difference appears as the quantity F deviates from 1.

in which $m \simeq 8\lambda(R/R_c)e^{-2R/R_c}$ in the region $R \gg R_c$. For increasing R the quantity m approaches $+0$ even faster than in the model (79). Another similar model is $f(R) = R - \lambda R_c(1 - e^{-R/R_c})$ [46], in which case $m \simeq \lambda(R/R_c)e^{-R/R_c}$ for $R \gg R_c$.

For the viable $f(R)$ models mentioned above the quantity F is close to 1 in the region $R \gg R_c$, so that the evolution of the horizon entropy S is similar to that of \hat{S} for the redshift $z \equiv a_0/a - 1 \gg 1$ (a_0 is the scale factor today). The deviation from the Λ CDM model appears for low redshifts ($z \lesssim 1$), which leads to the difference between S and \hat{S} . Since $f_{,RR} > 0$ for $R \geq R_1$, we have $\dot{F} < 0$ and hence $F < 1$ provided that R decreases with time ($\dot{R} < 0$). This means that \hat{S} should be smaller than S for low redshifts, which can be confirmed in the numerical simulation of Fig. 2 for the model (79).

In the following we study the evolution of S and \hat{S} in more details. The horizon entropy $S \propto H^{-2}$ in the equilibrium picture increases as long as H continues to decrease toward the de Sitter attractor. The stability of the de Sitter point given in Eq. (78) can be divided into two cases: (a) stable spiral for $0 < m(R_1) < 16/25$ and (b) stable node for $16/25 < m(R_1) < 1$ [34].¹

¹ This comes from the fact that the eigenvalues for the 3×3 matrix of perturbations about the de Sitter point are given by -3 ,

In the case (a) the solutions approach the attractor with the oscillation of R , whereas in the case (b) the oscillation of the Ricci scalar does not occur around $R = R_1$. In the former case the horizon entropy S finally approaches a constant value at the de Sitter point with small oscillations. The numerical simulation in Fig. 2 corresponds to this situation with $m(R_1) = 0.358$, which shows that the oscillation of S around the redshift $-1 < z < -0.8$ is really tiny. In the case (b) we have numerically checked that such oscillations of S disappear, as expected analytically. We have carried out numerical simulations for other viable $f(R)$ models such as (80) and found that the above properties also persist in those models. Thus the horizon entropy S globally increases with time apart from small oscillations that appear for the case $0 < m(R_1) < 16/25$.

Let us estimate the contribution of $d\hat{S}$ and $d_i\hat{S}$ to dS in Eq. (48). We shall consider the cosmological epoch in which the quantity \dot{H}/H^2 is approximately constant, that is, $\dot{H}/H^2 \simeq -(3/2)(1 + w_{\text{eff}})$, where w_{eff} is an effective equation of the system ($w_{\text{eff}} = 1/3, 0, -1$ during radiation, matter, and de Sitter eras, respectively). Then the last term in Eq. (48) is approximately given by

$$\overline{d_i\hat{S}} \equiv \frac{1}{F} \frac{2H^2 + \dot{H}}{4H^2 + \dot{H}} d_i\hat{S} \simeq -\frac{2\pi}{G} \frac{\dot{H}}{H^3} \frac{Rf_{,RR}}{f_{,R}} dt. \quad (81)$$

Using $dS = -(2\pi/G)(\dot{H}/H^3)dt$, the entropy production term can be simply expressed as

$$\overline{d_i\hat{S}} \simeq m dS. \quad (82)$$

It then follows from Eq. (48) that

$$d\hat{S}/F \simeq (1 - m) dS. \quad (83)$$

As long as $m \ll 1$ the entropy production term in Eq. (82) is negligible relative to dS , so that $dS \simeq (1/F)d\hat{S}$. As the deviation from the Λ CDM model appears, i.e. $m \gtrsim \mathcal{O}(0.1)$, the entropy production provides an important contribution to dS .

As long as the stability condition (78) is satisfied, the deviation parameter has been in the range $0 < m < 1$ until the solutions reach the de Sitter attractor. Provided that $dS > 0$, we then have $d_i\hat{S} > 0$ and $d\hat{S} > 0$ from Eqs. (82) and (83). In fact the growth of \hat{S} can be confirmed in Fig. 2, apart from the tiny oscillations around the de Sitter attractor.

VI. CONCLUSIONS

In the present paper, we have studied thermodynamics on the apparent horizon with area A in the FLRW

space-time for modified gravity theories with the Lagrangian density $f(R, \phi, X)$. If we define the energy momentum tensor of “dark” components other than perfect fluids as Eq. (13) with the Einstein equation (14), the corresponding energy density (11) and the pressure (12) do not satisfy the standard continuity equation for the theories in which the quantity $F = \partial f / \partial R$ is not constant. Introducing the Wald’s horizon entropy in the form $\hat{S} = AF/(4G)$ associated with a Noether charge, we have derived the first-law of thermodynamics given by Eq. (27) in the presence of a non-equilibrium entropy production term $d_i\hat{S}$. This non-equilibrium picture of thermodynamics arises for the theories with $dF \neq 0$, which include $f(R)$ gravity and scalar-tensor theories.

If we define the energy density ρ_d and the pressure P_d of dark components as in Eqs. (31) and (32) respectively, we obtain the standard continuity equation (33). This corresponds to the introduction of the energy momentum tensor $T_{\mu\nu}^{(d)}$ as Eq. (46) with the Einstein equation (45). Introducing the Bekenstein-Hawking entropy in the form $S = A/(4G)$, we have found that the first law of equilibrium thermodynamics follows from Einstein equations. Note that this is different from the approach taken in Ref. [13] for the realization of equilibrium thermodynamics in which a modified Misner-Sharp mass \mathcal{M} was introduced with the definition of the horizon entropy $\hat{S} = AF/(4G)$.

The horizon entropy S in our equilibrium framework is analogous to that in Einstein gravity. We note, however, that this equilibrium thermodynamics in the Jordan frame is not in general identical to that in the Einstein frame. For scalar-tensor theories with the action (50) we have derived the explicit relations between thermodynamical quantities in two frames. Our equilibrium description of thermodynamics in the Jordan frame is valid even for modified gravity theories in which the Einstein frame action does not exist. In addition the Jordan frame should be regarded as a physical one because of the conservation law of baryons. Hence the direct construction of equilibrium thermodynamics in the Jordan frame makes much more sense relative to that in the Einstein frame.

In the flat FLRW background the horizon entropy S is proportional to H^{-2} , which grows for decreasing H . In other words the violation of the null energy condition ($\rho_T + P_T \geq 0$, where ρ_T and P_T are the total energy density and the pressure respectively) can lead to the decrease of S . We have applied our formalism to $f(R)$ inflation models with the Lagrangian density $f(R) = R + \alpha R^n$ with $(\sqrt{3} + 1)/2 < n \leq 2$ and showed that S globally increases apart from the oscillation in the reheating phase. The global increase of S also persists in $f(R)$ dark energy models that satisfy cosmological and local gravity constraints in which the solutions approach a de Sitter attractor.

As we have derived in Eq. (48), the variation of S can be expressed in terms of $d\hat{S}$ in the non-equilibrium framework together with the entropy production term

$$-3/2 \pm \sqrt{25 - 16/m(R_1)}/2.$$

$d_i \hat{S}$. The last term in Eq. (48), which we denote $\overline{d_i \hat{S}}$, can be important for the models in which the quantity F departs from 1. In $f(R)$ dark energy models, for example, it follows that $\overline{d_i \hat{S}} \simeq m dS$ and $d\hat{S}/F \simeq (1-m)dS$, where $m = Rf_{,RR}/f_{,R}$ corresponds to the deviation parameter from the Λ CDM model. As long as $m \ll 1$ the entropy production term can be negligible such that $dS \simeq d\hat{S}/F$, but its contribution to dS becomes important for $m \gtrsim \mathcal{O}(0.1)$. The transition from the former to the latter regime indeed occurs at low redshifts for viable $f(R)$ dark energy models.

We have thus shown that the equilibrium description of thermodynamics in the Jordan frame is present for general modified gravity theories. The equilibrium description of thermodynamics is useful not only to provide the General Relativistic analogue of the horizon entropy irrespective of gravitational theories but also to understand the nonequilibrium thermodynamics deeper in connection with the standard equilibrium framework. It will be of interest to apply our formalism to dark energy dom-

inated universe by taking into account the entropies of dark energy as well as matter inside the horizon along the lines of Ref. [15].

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- [1] J. D. Bekenstein, Phys. Rev. D **7**, 2333 (1973).
 - [2] J. M. Bardeen, B. Carter and S. W. Hawking, Commun. Math. Phys. **31**, 161 (1973).
 - [3] S. W. Hawking, Commun. Math. Phys. **43**, 199 (1975).
 - [4] T. Jacobson, Phys. Rev. Lett. **75**, 1260 (1995).
 - [5] A. V. Frolov and L. Kofman, JCAP **0305**, 009 (2003).
 - [6] U. H. Danielsson, Phys. Rev. D **71**, 023516 (2005).
 - [7] R. Bousso, Phys. Rev. D **71**, 064024 (2005).
 - [8] S. A. Hayward, Phys. Rev. D **49**, 6467 (1994); Class. Quant. Grav. **15**, 3147 (1998); S. A. Hayward, S. Mukohyama and M. C. Ashworth, Phys. Lett. A **256**, 347 (1999).
 - [9] T. Padmanabhan, Class. Quant. Grav. **19**, 5387 (2002).
 - [10] M. Akbar and R. G. Cai, Phys. Rev. D **75**, 084003 (2007).
 - [11] C. Eling, R. Guedens and T. Jacobson, Phys. Rev. Lett. **96**, 121301 (2006).
 - [12] M. Akbar and R. G. Cai, Phys. Lett. B **635**, 7 (2006); Phys. Lett. B **648**, 243 (2007).
 - [13] Y. Gong and A. Wang, Phys. Rev. Lett. **99**, 211301 (2007).
 - [14] S. F. Wu, B. Wang and G. H. Yang, Nucl. Phys. B **799**, 330 (2008).
 - [15] S. F. Wu, B. Wang, G. H. Yang and P. M. Zhang, Class. Quant. Grav. **25**, 235018 (2008).
 - [16] K. Bamba and C. Q. Geng, Phys. Lett. B **679**, 282 (2009).
 - [17] R. G. Cai and L. M. Cao, Phys. Rev. D **75**, 064008 (2007).
 - [18] A. Paranjape, S. Sarkar and T. Padmanabhan, Phys. Rev. D **74**, 104015 (2006).
 - [19] R. G. Cai, L. M. Cao, Y. P. Hu and S. P. Kim, Phys. Rev. D **78**, 124012 (2008).
 - [20] A. Sheykhi, B. Wang and R. G. Cai, Nucl. Phys. B **779**, 1 (2007); Phys. Rev. D **76**, 023515 (2007); X. H. Ge, Phys. Lett. B **651**, 49 (2007); S. F. Wu, G. H. Yang and P. M. Zhang, arXiv:0710.5394 [hep-th].
 - [21] R. G. Cai and S. P. Kim, JHEP **0502**, 050 (2005).
 - [22] R. M. Wald, Phys. Rev. D **48**, 3427 (1993).
 - [23] V. Iyer and R. M. Wald, Phys. Rev. D **50**, 846 (1994).
 - [24] J. c. Hwang and H. Noh, Phys. Rev. D **71**, 063536 (2005); S. Tsujikawa, Phys. Rev. D **76**, 023514 (2007).
 - [25] C. W. Misner and D. H. Sharp, Phys. Rev. **136**, B571 (1964).
 - [26] R. Brustein, D. Gorbonos and M. Hadad, Phys. Rev. D **79**, 044025 (2009).
 - [27] K. i. Maeda, Phys. Rev. D **39**, 3159 (1989).
 - [28] S. Tsujikawa, K. Uddin, S. Mizuno, R. Tavakol and J. Yokoyama, Phys. Rev. D **77**, 103009 (2008).
 - [29] L. Amendola, Phys. Rev. D **62**, 043511 (2000).
 - [30] A. A. Starobinsky, Phys. Lett. B **91**, 99 (1980).
 - [31] M. B. Mijic, M. S. Morris and W. M. Suen, Phys. Rev. D **34**, 2934 (1986).
 - [32] V. Muller, H. J. Schmidt and A. A. Starobinsky, Phys. Lett. B **202**, 198 (1988).
 - [33] V. Faraoni, Phys. Rev. D **72**, 124005 (2005).
 - [34] L. Amendola, R. Gannouji, D. Polarski and S. Tsujikawa, Phys. Rev. D **75**, 083504 (2007).
 - [35] S. M. Carroll, I. Sawicki, A. Silvestri and M. Trodden, New J. Phys. **8**, 323 (2006); T. Faulkner, M. Tegmark, E. F. Bunn and Y. Mao, Phys. Rev. D **76**, 063505 (2007); Y. S. Song, W. Hu and I. Sawicki, Phys. Rev. D **75**, 044004 (2007); R. Bean, D. Bernat, L. Pogosian, A. Silvestri and M. Trodden, Phys. Rev. D **75**, 064020 (2007); Y. S. Song, H. Peiris and W. Hu, Phys. Rev. D **76**, 063517 (2007); L. Pogosian and A. Silvestri, Phys. Rev. D **77**, 023503 (2008).
 - [36] L. Amendola, D. Polarski and S. Tsujikawa, Phys. Rev. Lett. **98**, 131302 (2007); Int. J. Mod. Phys. D **16**, 1555 (2007).
 - [37] G. J. Olmo, Phys. Rev. D **72**, 083505 (2005); A. L. Erickcek, T. L. Smith and M. Kamionkowski, Phys. Rev. D **74**, 121501 (2006); V. Faraoni, Phys. Rev. D **74**, 023529 (2006); I. Navarro and K. Van Acoleyen, JCAP **0702**, 022 (2007); T. Chiba, T. L. Smith and A. L. Erickcek, Phys. Rev. D **75**, 124014 (2007); P. Brax, C. van de Bruck, A. C. Davis and D. J. Shaw, Phys. Rev. D **78**,

- 104021 (2008); I. Thongkool, M. Sami, R. Gannouji and S. Jhingan, Phys. Rev. D **80**, 043523 (2009).
- [38] S. Tsujikawa, R. Gannouji, B. Moraes and D. Polarski, Phys. Rev. D **80**, 084044 (2009).
- [39] B. Li and J. D. Barrow, Phys. Rev. D **75**, 084010 (2007).
- [40] L. Amendola and S. Tsujikawa, Phys. Lett. B **660**, 125 (2008).
- [41] W. Hu and I. Sawicki, Phys. Rev. D **76**, 064004 (2007).
- [42] A. A. Starobinsky, JETP Lett. **86**, 157 (2007).
- [43] S. A. Appleby and R. A. Battye, Phys. Lett. B **654**, 7 (2007).
- [44] S. Tsujikawa, Phys. Rev. D **77**, 023507 (2008).
- [45] S. Nojiri and S. D. Odintsov, Phys. Lett. B **657**, 238 (2007).
- [46] E. V. Linder, Phys. Rev. D **80**, 123528 (2009).
- [47] S. Capozziello and S. Tsujikawa, Phys. Rev. D **77**, 107501 (2008).